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EGYPTIAN MATHEMATICS

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Ivor Grattan-Guinness in memoriam

The primary references of the term "Egyptian mathematics" are the computational techniques and the underlying mathematical knowledge attested in Pharaonic written sources. Secondary references are, on one hand, the corresponding techniques etc. as known from demotic sources and, on the other, the geometrical procedures used in Pharaonic and subsequent architecture and visual arts. Greek mathematics produced in Hellenistic Egypt is thus *not* included. Accordingly, all dates in the following are BCE when not specified to be CE.

THE SOURCES

The most important written sources for Pharaonic mathematics are the *Rhind Mathematical Papyrus* (henceforth RMP)[1] and the *Moscow Mathematical*

[1] Two editions with ample analysis exist: T. Eric Peet, *The Rhind Mathematical Papyrus, British Museum 10057 and 10058. Introduction, Transcription, Translation and Commentary* (London: University Press of Liverpool, 1923); Arnold Buffum Chace, *The Rhind Mathematical Papyrus. British Museum 10057 and 10058*, 2 vols. (Oberlin, OH: Mathematical Association of America, 1927–9), vol. 1 (with the assistance of Henry Parker Manning): *Free Translation and Commentary and a Bibliography of Egyptian Mathematics by R. C. Archibald*; vol. 2 (with Ludlow Bull and Henry Parker Manning): *Photographs, Transcription, Transliteration, Literal Translation and a Bibliography of Egyptian and Babylonian Mathematics (Supplement), by R. C. Archibald and the Mathematical Leather Roll in the British Museum, by S. R. K. Glanville* (the literal translation is indeed *very* literal; it is used in all quotations below from RMP). A recent facsimile edition with description and discussion and translation of large extracts is Gay Robins and Charles Shute, *The Rhind Mathematical Papyrus: An Ancient Egyptian Text* (London: British Museum Publications, 1987).Valuable general accounts of Egyptian mathematics which (by necessity) deal extensively with RMP are: Kurt Vogel, *Vorgriechische Mathematik.* Volume 1*: Vorgeschichte und Ägypten* (Mathematische Studienhefte, 1; Hanover: Hermann Schroedel / Paderborn: Ferdinand Schöningh, 1958); Richard J. Gillings, *Mathematics in the Time of the Pharaohs* (Cambridge, MA: MIT Press, 1972). After the manuscript for the present article was finished (essentially in 1997), Marshall Clagett has published *Ancient Egyptian Science. A Source Book.* Volume 3*: Ancient Egyptian Mathematics* (Memoirs of the American Philosophical Society, 232; Philadelphia, PA: American Philosophical Society, 1999). This volume reproduces the facsimile edition of Chace's volume 2 and includes a translation kept close to that of the same volume as well as an extensive analysis. Extensive analysis of RMP as well as other Middle Kingdom sources is also found in Annette Imhausen, *Ägyptische Algorithmen. Eine Untersuchung zu den mitteläg yptischen mathematischen Aufgabentexten* (Ägyptologische Abhandlungen, 65; Wiesbaden: Harrassowitz, 2003). Egyptian mathematics in its social setting is dealt

Papyrus (MMP).[2] To these can be added a few shorter papyri containing mathematical problems; a couple of listings of equivalent fractions; and a larger number of accounting papyri that apply the metrology and show how the basic arithmetical techniques were used. The RMP is a copy from a Middle Kingdom original (Amenemhet III, c. 1800), made during the Hyksos period; it is a teacher's or calculator's manual, containing several tables (to which we shall return) and some eighty problems with solution. The MMP seems to be a late Middle Kingdom copy from an earlier Middle Kingdom original; it is a collection of student's answers to problems, provided with the teacher's approval (at times justly refused). The other properly mathematical sources are from the Middle through New Kingdom.[3]

A source of particular character is the New Kingdom fictional "satirical letter" or Papyrus Anastasi I,[4] in which a scribe chides a colleague for his professional ignorance; it shows that a military scribe was supposed to be familiar with Palestinian geography and with the determination of the manpower, rations, and other requirements of construction work. Accounting papyri show that other categories of scribes had analogous tasks.

Some administrative records go back to the Old Kingdom (and a few documents containing numbers to the early dynastic period); even in this respect, however, the Middle through New Kingdom is much richer.

The mathematical sources of the Pharaonic period are written in hieratic script (evidently, some hieroglyphic documents contain numbers); part of the metrological terminology seems to have been created in hieratic and to have acquired hieroglyphic equivalents only at a later moment (when at all).

with in Annette Imhausen, *Mathematics in Ancient Egypt: A Contextual History* (Princeton & Oxford: Princeton University Press, 2016).

2 Edition with translation and extensive commentary: W. W. Struve, *Mathematischer Papyrus des Staatlichen Museums der Schönen Künste in Moskau* (Quellen und Studien zur Geschichte der Mathematik. Abteilung A: Quellen, 1. Band; Berlin: Julius Springer, 1930). The edition is reproduced in Marshall Clagett, *Ancient Egyptian Science*, vol. 3, which also contains an English translation and a commentary.

3 Marshall Clagett, following recent research, gives the following approximate dates (*Ancient Egyptian Science. A Source Book.* Volume I*: Knowledge and Order* (Memoirs of the American Philosophical Society, 184 A+B; Philadelphia, PA: American Philosophical Society, 1989), pp. 629–35):

Early dynastic period (dynasties 1–2): 3110–2665.
Old Kingdom (dynasties 3–8): 2664–2155.
First intermediate period (dynasties 9–10): 2154–2052.
Middle Kingdom (dynasties 11–13): 2040–1640.
Second intermediate period (Hyksos dynasties 15–16, Theban dynasty 17): 1640–1532.
New Kingdom (dynasties 18–20): 1550–1070.
Third intermediate period (dynasties 21–(initial) 25): 1070–712.
Late period (dynasties (final) 25–31, including the Assyrian hegemony during dynasty 26 and the Persian dynasties 27 and 31): 712–332.
Greco-Roman period: 332 BCE to 395 CE.

4 Full edition and translation in Alan H. Gardiner, *Egyptian Hieratic Texts.* Series I*: Literary Texts from the New Kingdom.* Part I*: The Papyrus Anastasi I and the Papyrus Koller, together with Parallel Texts* (Leipzig: J. C. Hinrichs'sche Buchhandlung, 1911).

From the demotic phase (late and Greco-Roman periods), several papyri containing mathematical problems and tables survive[5] – with the possible exception of one uncertain fragment all of Greco-Roman date.

NUMBERS AND METROLOGY

The Egyptian number system was decadic. Already in early dynastic times, individual signs for 1, 10, 100, 1,000, 10,000, 100,000, and 1,000,000 existed.[6] In hieroglyphic, other numbers were constructed additively, by mere repetition of these signs; in hieratic, these were contracted into individual signs for 1, 2, . . ., 9, 10, 20, . . ., 90, 100, 200, etc. From the Middle Kingdom onward, fractional numbers were expressed as sums of aliquot parts (including two-thirds); in order to keep close to the Egyptian notation we may transcribe them as follows: $3''$ ($= {}^2/_3$), $2'$ ($= {}^1/_2$), $3'$ ($= {}^1/_3$), $\overline{4}$ ($= {}^1/_4$), $\overline{5}$, etc. ($3''$, $2'$, and $3'$ had special signs); the others were denoted by the sign *ro* ("mouth," here "part") or by a dot above the number (in hieroglyphic respectively hieratic writing), according to a canon that did not allow repetition of the same aliquot part but expressed for instance $\overline{5}\ \overline{5}$ as $3'$ $1\overline{5}$ (juxtaposition means addition). Essential metrologies, however, would operate with subunits instead of these fractions.

Many of the problems in the mathematical texts deal with the difficulties to which the non-decadic metrologies would give rise. Closest to decadic principles is the length system. The basic length unit was the "royal cubit" (*mḥ*, c. 52cm), subdivided into 7 "palms" of 4 "fingers" each (a "short cubit" of 6 palms was also in use). 100 royal cubits was a "rope" (*khet*). Land might be measured in *setat* (that is, square *khet*), divided by successive halvings into subunits with special names (down to ${}^1/_{32}$ *setat*); in surveying practice, the "cubit of land" (1 cubit versus 1 *khet*) and the "thousand of land" (1,000 cubit versus 1 *khet*) were mostly preferred.

The central capacity unit was the *ḥeḳat*, divided according to one system into 10 *henu* or 320 *ro* (the "part" again, but in a different use), according to another by successive halvings (down to ${}^1/_{64}$).[7] Multiples of the *ḥeḳat* were expressed by special signs or by non-standard use of the standard numerals. A special unit for bulky substances is the *khar*, equal to 20 *ḥeḳat* and to ${}^2/_3$ of a cubic royal cubit (probably a secondary normalization of originally distinct units).

[5] Edition with translation and extensive commentary in Richard A. Parker (ed.), *Demotic Mathematical Papyri* (Providence, RI and London: Brown University Press, 1972).

[6] The standard reference for Egyptian numerals and number words remains Kurt Sethe, *Von Zahlen und Zahlworten bei den Alten Ägyptern, und was für andere Völker und Sprachen daraus zu lernen ist* (Schriften der Wissenschaftlichen Gesellschaft in Straßburg, 25; Straßburg: Karl J. Trübner, 1916). In the second millennium, the sign for 1,000,000, and afterwards that for 100,000, went out of use; instead, multiplicative notations were used.

[7] The hieroglyphic signs for the successive halves of the *ḥeḳat* can be put together to the standard representation of the healing sacred eye of Horus, as pointed out by Peet, *The Rhind Mathematical Papyrus*, p. 26. However, the hieroglyphic writings do not antedate the Eighteenth Dynasty, whereas the hieratic forms go back to the third millennium; the mythological connotations of the system are

BASIC PATTERNS AND TECHNIQUES

Egyptian arithmetical thinking may be interpreted as based on two key principles: additivity and proportionality – the latter in the sense that any number might count another number; to this come the techniques of doubling and multiplying by 10. The multiplication of 75 by 53 might be performed thus:

/1	75
/2	150
/10	750
20	1500
/40	3000
Total	3975

Some texts reveal the underlying thought: If *1* (of the entity we count) is 75, then *2* (of it) is 150, etc. The multiplier 53, as we see, is split into components that can be obtained by successive doublings and decouplings (mostly, only doublings would be employed). Strokes mark addends that are actually used (53 = 1+2+10+40).

The corresponding division of 3975 by 75 would go by the same procedure, emptying 3975 by multiples of 75:

/1	75
/10	750
20	1500
/40	3000
/2	150
Total	3975

A separate phrase would state the result as 53 (= 1+10+40+2); strokes will of course have been inserted a posteriori in the scheme.

This remains simple only until fractions are introduced. An actual multiplication (of 8 3″ $\overline{6}$ $\overline{18}$ = $8^8/_9$ by itself) would run as follows (RMP 42):

1	8 3″ $\overline{6}$ $\overline{18}$
2	17 3″ $\overline{9}$
4	35 2′ $\overline{18}$
/8	71 $\overline{9}$
/3″	5 3″ $\overline{6}$ $\overline{18}$ $\overline{27}$
3′	2 3″ $\overline{6}$ $\overline{12}$ $\overline{36}$ $\overline{54}$
/$\overline{6}$	1 3′ $\overline{12}$ $\overline{24}$ $\overline{72}$ $\overline{108}$
/$\overline{18}$	3′ $\overline{9}$ $\overline{27}$ $\overline{108}$ $\overline{324}$
Total	79 $\overline{108}$ $\overline{324}$

It is no accident that 3″ of 8 3″ $\overline{6}$ $1\overline{8}$ is found before 3′. Even when only 3′ of the multiplicand is needed, 3″ is found first and 3′ then by halving. 3″ and 2′ were the basic fractions of the Middle Kingdom calculators; if at all possible, further divisions would be produced from these by successive halvings (the presence of $1\overline{8}$ illustrates that it was not always possible).

Beyond this, the calculation displays the main difficulties to which multiplication of fractions gives rise. The first doubling is obvious, since 3″ doubled is 1 3′; in the next, however, $\overline{9}$ has to be doubled, and the scribe has to know that this yields $\overline{6}$ $1\overline{8}$ (after which 3′ $\overline{6}$ is contracted to 2′). Finally, $\overline{9}$ 3″ $\overline{6}$ $1\overline{8}$ $2\overline{7}$ 3′ $1\overline{2}$ $2\overline{4}$ $7\overline{2}$ $10\overline{8}$ 3’ $\overline{9}$ $2\overline{7}$ $10\overline{8}$ $32\overline{4}$ has to be converted into 2 $10\overline{8}$ $32\overline{4}$.

For the former purpose, RMP contains a tabulation of 2÷*n*, for all odd values of *n* from 5 to 101. For the latter, a technique referred to as “red auxiliary numbers” was used. The fractions might be expressed as fractions of an adequate “reference magnitude” – in the present case probably 108 – in a scheme (red is rendered by italics):

$\bar{9}$	3″	$\bar{6}$	$1\bar{8}$	$2\bar{7}$	3′	$1\bar{2}$	$2\bar{4}$	$7\bar{2}$	$10\bar{8}$	3′	$\bar{9}$	$2\bar{7}$	$10\bar{8}$	$32\bar{4}$
12	*72*	*18*	*6*	*4*	*36*	*9*	*4 $\bar{2}$*	*1 $\bar{2}$*	*1*	*36*	*12*	*4*	*1*	*3′*

Since the sum of the red (i.e., italicized) numbers is 217 3′= 2·108+1+3′, the sum of the fractions is 2+$10\overline{8}$ +$32\overline{4}$. Structurally, this is equivalent to the use of a common denominator 108, and there are some hints that a notion of the fraction $^{p}/_{q}$ understood as *p* copies of $\overline{q}$ was not as strange to Egyptian calculators as the stylistic canon might make us believe – in RMP 81, the scribe erroneously writes $\overline{5}$ and 3 instead of $\overline{2}$ $\overline{8}$ (= $^{5}/_{8}$) and $\overline{4}$ $\overline{8}$ (= $^{3}/_{8}$).[8] Nonetheless, an interpretation of the underlying thought in terms of a reference magnitude[9] agrees so well with the global pattern of the texts that it is likely to be the primary explanation of the red auxiliaries.

/1	7
/$\overline{7}$	1
1	8
/2	16
2′	4
/$\overline{4}$	2
/$\overline{8}$	1
/1	2 $\overline{4}$ $\overline{8}$
/2	4 2′ $\overline{4}$
/4	9 2′

8 This was first pointed out by Kurt Vogel in *Die Grundlagen der ägyptischen Arithmetik* (first published Munich, 1929; reprint Wiesbaden: Martin Sändig, 1970), p. 43.

9 First proposed in Léon Rodet, “Les prétendus problèmes d’algèbre du manuel du calculateur égyptien (Papyrus Rhind),” *Journal asiatique*, septième série 18 (1881), 184–232, 390–559.

Most everyday practical computation above the level of counting is based on proportionality in one or the other way – since the Middle Ages often in the shape of the Rule of Three. The Egyptian approach may be illustrated by RMP 24, one of the problems treating of an abstract "quantity" or "heap" ^{c}*ḥ*^{c} – the problem type by which the technique was trained: "A quantity, $\overline{7}$ of it added to it, becomes it: 19."

The computation looks as follows

/1	7	The doing as it occurs.	
/$\bar{7}$	1	The quantity	16 2′ $\bar{8}$
		$\bar{7}$	2 $\bar{4}$ $\bar{8}$
1	8	Total	19
/2	16		
2′	4		
/$\bar{4}$	2		
/$\bar{8}$	1		
/1	2 $\bar{4}$ $\bar{8}$		
/2	4 2′ $\bar{4}$		
/4	9 2′		

This may be explained as a "single false position": as a preliminary value for the heap we take 7; then the quantity together with its seventh part becomes 8. This is seen to be contained 2 $\overline{4}$ $\overline{8}$ times in 19 (an ordinary division); therefore the true value of the quantity is 2 $\overline{4}$ $\overline{8}$ times 7 – or, which is more convenient for the final proof, 7 times 2 $\overline{4}$ $\overline{8}$ = 16 2′ $\overline{8}$. The Egyptians, indeed, made ample use of the commutativity of multiplication, despite the obvious asymmetry of their algorithm; the frequent claim that their mathematical thought was purely additive is thus blatantly mistaken.

The principles of this computation were applied with flexibility: at times the preliminary value might be set to *1* (e.g. RMP 32); in combination with the commutativity of operations this might lead to something very close to the Babylonian division through multiplication by the reciprocal (e.g. RMP 63). The formulations, however, show that the Egyptian method is based on the usual principles and no borrowing from abroad.

The $2{\div}n$ table of RMP is the largest extant piece of systematic Egyptian mathematics and may be considered its theoretical high point. Much effort has hence been dedicated to finding the principle(s) which underlie its construction – the same fraction may indeed be split in many different ways into aliquot parts ($^{2}/_{15}$ thus into $\overline{8}$ $\overline{120}$, $\overline{9}$ $\overline{45}$,

$\overline{10}$ $\overline{30}$, $1\overline{2}$ $\overline{20}$, $\overline{11}$ $\overline{30}$ $\overline{110}$, $\overline{13}$ $\overline{20}$ $\overline{156}$, $\overline{14}$ $\overline{30}$ $\overline{35}$, etc.). So much is certain that a standard existed in the later Middle Kingdom – the deviations from the RMP-norm are rare enough to count as aberrations. Kurt Vogel points to three principles (at times in mutual conflict) that seem to intervene:[10]

(i) The members of the sum should be few.
(ii) The first member should be as large as possible.
(iii) If more than two members are present, the largest denominator should be kept small.

(ii) might seem to suggest a search for a good first approximation – but (iii) shows that a good second approximation was not aimed at. The principles seem rather to have been of an aesthetic kind.

The technique that is used consists in dividing 2 into two parts $p+r$, where p is an aliquot part of n ($^{p}/_{n} = {}^{1}/_{m}$) and the remainder r is an aliquot part of 1 or the sum of such parts, $r = {}^{1}/_{s} + {}^{1}/_{t} + \ldots$, whence $^{r}/_{n} = {}^{1}/_{(sn)} + {}^{1}/_{(tn)} + \ldots$[11] This much is shown explicitly in the text, which lists p and the constituents of r and tells which part ($^{1}/_{m}$, $^{1}/_{(sn)}$, etc.) each one is of n. The essential trick, however, is of course to find an adequate splitting of 2. Here several ways were followed, perhaps reflecting the steps of the historical process that had engendered the table. If n is a multiple of 3 ($n = 3\,k$), the division is into 1 2′ and 2′, whence $2 \div n = {}^{1}/_{2\,k} + {}^{1}/_{6\,k}$. In many other cases, an adequate p was probably found by subdivision of 3" of n or 2' of n, as illustrated by the way the text explains 2÷13:

	1	13	*$\overline{8}$* 1 2′ $\overline{8}$ *$5\overline{2}$* $\overline{4}$ *$10\overline{4}$* $\overline{8}$
	2′	6 2′	
	$\overline{4}$	3 $\overline{4}$	
	/$\overline{8}$	1 2′ $\overline{8}$	
/4	$5\overline{2}$	$\overline{4}$	
/8	$10\overline{8}$	$\overline{8}$	

At first, 13 is subdivided by successive halvings until we get below 2; then 1 2′ $\overline{4}$ (= $\overline{8}$ of 13) is chosen as p, and the remainder is seen to consist of $\overline{4}$ (= $\overline{s}$) and $\overline{8}$ (= $\overline{t}$). n = 13 (considered as a "weak sign," i.e., as the representative of an aliquot

[10] Vogel, *Vorgriechische Mathematik*, vol. 1, 42.

[11] In two cases, n = 35 and n = 91, r/n turns out to be an aliquot part even though r iself is composite ($2 \div 35 = \overline{30} + \overline{42}$, $2 \div 91 = \overline{70} + \overline{130}$).

part[12]) is then multiplied by 4 and 8, and we see that the *numbers* $\overline{4}$ and $\overline{8}$ are 5$\overline{2}$ and 10$\overline{4}$ *of 13*. The summary in the right column tells that 2 is $\overline{8}$ 5$\overline{2}$ 10$\overline{4}$ of 13.

In other cases m is stated directly, often as one of the abundant numbers 30 and 60. It cannot be excluded that these choices resulted from mere trial and error – values of m with a profusion of divisors are most likely to permit a nice splitting of the remainder r – but it seems more plausible that the Egyptians had discovered that 30 and 60 are often convenient choices and took this as their first guess;[13] however that may be, the procedure makes use of a reference magnitude or of splitting into smaller parts. We may look at 2÷73:

	73	*6$\overline{0}$* 1 $\overline{6}$ 2$\overline{0}$	*21$\overline{9}$* 3′	*29$\overline{2}$* $\overline{4}$	*36$\overline{5}$* $\overline{5}$
Find	\6$\overline{0}$	1 $\overline{6}$ 2$\overline{0}$			
\3	21$\overline{9}$	3′			
\4	29$\overline{2}$	$\overline{4}$			
\5	36$\overline{5}$	$\overline{5}$			

This can be understood as follows (structurally equivalent interpretations are possible): 2 is split into 120 parts, each of which is then 6$\overline{0}$. 73 of these divided by 73 make 6$\overline{0}$; since 73 = 60+10+3, 73÷60 = 1 $\overline{6}$ 2$\overline{0}$ = p. The right-hand column tells that the remainder until 2 after the removal of p is 3′ $\overline{4}$ $\overline{5}$ (namely 47 of the 120 small parts of 2, grouped as 20+15+12; possible alternatives are 30+15+2 and 30+12+5 – the actual choice illustrates Vogel's rule (iii)); multiplication of 73 (considered "weak") by 3, 4, and 5 (left-hand column), respectively, shows that the remainder is 21$\overline{9}$ 29$\overline{2}$ 36$\overline{5}$ of 73 (middle column); all is summarized in the first line.

APPLIED ARITHMETIC

Beyond the abstract $^{c}\underset{\cdot}{h}^{c}$-problems, both RMP and MMP contain many arithmetical problems of practical or sham-practical character. Most important are distribution problems and the so-called *pesu*-problems.

Many of the distribution problems deal with equal partition – e.g., the distribution of n loaves among 10 persons, n = 1, 2, 6, 7, 8, 9 (RMP, 1–6); they illustrate why Plutarch and other Greek authors would link social equality to "arithmetical justice" (and hence reject the latter as morally unsound). Others follow the principle that the foreman and other officials get double share (RMP 65), or that the ratio between shares is given (RMP 63). Such problems are true to real life as revealed in administrative texts. RMP 40, on

12 This terminology is used in RMP 61B; cf. Peet, *The Rhind Mathematical Papyrus*, p. 104.

13 The repeated occurrence of 30 allows us to discard the hypothesis that the frequent choice of 60 was inspired by the Babylonian sexagesimal system.

the contrary, is wholly artificial: loaves are distributed in five shares (say, *a, b, c, d*, and *e*) in arithmetical proportion in such a way that

(i) $^{1}/_{7}$ of the sum of the three major shares equals the sum of the two minor ones;
(ii) $a+b+c+d+e = 100$.

The solution makes use of a simple false position: at first an arithmetical progression $\alpha, \beta, \gamma, \delta, \varepsilon$ is constructed, starting with $\alpha = 1$ and fulfilling (i); its sum is found to be 60, whence all members are multiplied by 1 3″. The first step is not explained, but since RMP 64 refers explicitly to and makes adequate use of the *average share* and the *excess* of one share over the other when determining the single members of an arithmetical progression from the sum and the difference, a simple algebraic solution (whether represented by words, by strokes on an ostrakon or by pebbles or other material tokens) will not have exceeded the conceptual capabilities of the Egyptian calculator, though apparently his standard discourse: if α is 1 and τ the difference, $\beta = 1+\tau$, $\gamma = 1+2\tau$, etc.; the sum of the three major shares is thus $3+(2+3+4)\tau$, which is 7 times $\alpha+\beta = 2+\tau$; thus $3+9\tau = 14+7\tau$, $2\tau = 11$, $\tau = 5\ 2'$.

Endowed with particular status – obviously because of the importance of bread and beer as staple food – are the *pesu* (*pfśw*) problems. *Pfśw* is derived from *pśj*, "cooking," and may be understood as "baking ratio." The *pesu* of a loaf is the number of similar loaves that may be made from one *ḥeḳat* of grain; similarly, the *pesu* of beer counts the number of jugs that are produced from one *ḥeḳat* of grain. In both cases, the baking ratio thus indicates the reciprocal grain content of the unit of consumption. *Pesu* problems may ask for the *pesu* given the number of units produced and the total amount of grain, or for the exchange of loaves with different *pesu* or of bread with beer. More complex problems deal with the dilution of beer, or with special brews made from several grain sorts or fruits.[14]

Together, these and other problems of applied arithmetic cover more or less the standard types of late medieval and early modern commercial arithmetic – proportional partition, exchange, alloying (only composite interest has no Egyptian counterpart); often the methods are familiar, although no technique similar to the double false position is ever applied; at times, however, unexpected steps demonstrate that ad hoc reasoning was no less important than automatic routines.

Among higher arithmetical problems, one recreational problem in RMP deals with the geometrical progression $7, 7^2, 7^3, 7^4, 7^5$, and finds the sum as $7 \cdot 2801$; nothing in the text tells whether the underlying reasoning is simply that $7+ \ldots +7^5 = 7 \cdot (1+ \ldots +7^4) = 7 \cdot 2801$, or a formula for the sum of a geometrical progression was known.[15]

[14] See Peet, *The Rhind Mathematical Papyrus*, pp. 112–21, and Struve, *Mathematischer Papyrus*, pp. 44–101.

[15] A third, somehow intermediate possibility, is suggested by Robins and Shute, *The Rhind Mathematical Papyrus*, pp. 56 f.

GEOMETRICAL COMPUTATION

Geometrical problems deal with slopes, areas, and volumes. The batter (*śd*) of pyramids is expressed as the retrocession in palms per cubit height, whereas that of a different (unidentified) structure is given as a pure-number ratio in RMP 60.

Already the metrology (cf. above section, "Numbers and Metrology") shows that rectangular areas were found as length times breadth. The area of a triangle was determined as half the base multiplied by "the edge," whose identity has been discussed; however, since RMP 51 takes the half of the base "for the giving of the rectangle of it," there can be little doubt that the edge between the two parts into which an isosceles triangle is cut is meant – that is, the height.[16] The area of the trapezium was found correspondingly.

Area computation serves in a few cases as the basis for homogeneous second-degree problems. Thus in MMP 6, 7, and 17, the area of a right triangle and the ratio between the sides is given; doubling of the area and multiplication by the ratio yield the area of a corresponding square, whose square root ("corner") is then one side of the triangle; similar considerations are used to solve problems about two squares whose sides have a given ratio.

The volume of a right parallelepiped was found by multiplication of the three dimensions measured in cubits, followed by a multiplication by 1 2′ in order to express it in *khar*. MMP 14 finds the volume of a truncated square pyramid (with height h and sides a and b of base and top, respectively) correctly as ${}^{h}/_{3}\cdot(a^2+ab+b^2)$. No cues are given as to how the formula was derived. It cannot be excluded that it is the result of a lucky generalization of the formula for the area of a triangle; nor is a heuristic argument based on dissection into simpler volumes to be excluded, however.[17]

The area of the circle was found as that of the square on ${}^{8}/_{9}$ of the diameter – 1.006 . . . times the true value. A diagram in RMP 48 suggests that this may be a computational approximation to the area of a geometrically approximating octagon, whose area is ${}^{63}/_{81}$ of the square in question (see Figure 8.1.). Volumes of circular cylinders were determined accordingly.

MMP 10 calculates the surface of a "basket" with "mouth" 4 2′ as $4^{1}/_{2}\cdot({}^{8}/_{9}\cdot[{}^{8}/_{9}\cdot 9])$, with the argument that the "basket" is the half of an "egg" (Struve's reading of a damaged word). The double factor ${}^{8}/_{9}$ leaves no doubt that explicit use is made of the formula for the circular area – no empirical measurement would be able to distinguish $({}^{8}/_{9}\cdot[{}^{8}/_{9}\cdot 9]) = 7\ \overline{9}$ from 7 – and the conjectured "egg" seems to suggest that a hemisphere with diameter 4 2′ is intended, whose surface (in modern

16 This was already argued by Peet, *The Rhind Mathematical Papyrus*, pp. 91–3.

17 See, e.g., Robins and Shute, *The Rhind Mathematical Papyrus*, p. 49.

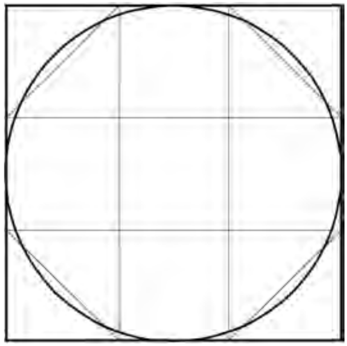

Figure 8.1. Method for finding the area of a circle.

terms) is then found correctly as $2\pi r^2$. This formula seems much more sophisticated that anything else found in the sources, for which reason the alternative interpretation of the "basket" as the curved surface of a semicylinder (with height = diameter = 4 2′) has been suggested. This does not fit an "egg" too well, but has the advantage to presuppose only that the Egyptians knew the relation between circular area and circumference – which agrees well with their explicit transformation of a triangle into a corresponding rectangle.[18]

GEOMETRICAL TECHNIQUES

Rules for geometrical computation evidently depend on techniques for mensuration. These will have been the responsibility of those "rope stretchers" (*harpedonaptai*) which the Greeks refer to.[19] Rope constructions were also used when the ground plans of prestige buildings were laid out. Architectural designs as well as pictorial art were constructed within square grids, following a strict canon (the "canonical system," coupled to metrology and already used in Early Dynastic iconography) for how many grid parts each part of a human body should occupy in the picture.[20]

[18] The two interpretations (due, respectively, to Struve and Peet) are confronted in grammatical detail in O. Neugebauer, *Vorgriechische Mathematik* (Berlin: Julius Springer, 1934), pp. 129–37.

[19] See Peet, *The Rhind Mathematical Papyrus*, p. 32, and Vogel, *Vorgriechische Mathematik*, vol. 1, 59 f.

[20] See Erik Iversen, *Canon and Proportion in Egyptian Art* (2nd edn; Warminster: Aris & Phillips, 1975) (first published 1955) and (on the influence of the system in later art) Erik Iversen, "The Canonical Tradition," in J. R. Harris (ed.), *The Legacy of Egypt* (2nd edn; Oxford: Oxford University Press, 1971), pp. 55–82. Corinna Rossi's *Architecture and Mathematics in Ancient Egypt* (Cambridge: Cambridge University Press, 2003) is a recommendable (and quite cautious) investigation of what can be said in general about the intertwinement of the two fields.

FABLES

Two remarks should be added concerning what we have *no* reason to ascribe to Pharaonic Egyptian geometry.

Firstly, ever since Moritz Cantor proposed that the rope stretchers *might* have used the 3-4-5-triangle to construct right triangles it has been a recurrent claim that they actually did so.[21] It has also been presumed that since several pyramids have the batter 3:4, the Egyptians will have known the properties of this triangle.[22] It must be emphasized that the sources do not contain the slightest hint pointing in this direction, and that the batter in question would be expressed as $5\ \overline{4}$ palms [per cubit height], which is not liable to have furthered any "Pythagorean" speculations.

Similarly, the attempts to find π (or the Golden Section) in the great pyramids founder on the observation that the approximate occurrence of such ratios in the construction are automatic consequences of the simple value for the batter; we should also remember that the Egyptians did not make use of π, that is, of the ratio between circular circumference and diameter, but of $\sqrt{\pi/4}$ (the ratio between the side of the squared circle and the diameter), which they approximated as $^{8}/_{9}$.[23]

ORIGINS AND DEVELOPMENT

The canonical system of proportions goes back to Dynasty 1; recordings of the Nile height in cubits, palms, and fingers (which probably served to fix the taxation level for the year) are roughly contemporary; biennial "countings" of the resources of the country begin with Dynasty 2; "chord stretching" at the foundation of prestige buildings is recorded during Dynasty 1.[24] Though nothing comparable to the bureaucratic precision of early Mesopotamian state formation was aimed at – the legitimization of the Pharaonic state rested on conquest and perhaps on the affirmation of cosmological stability, not on redistribution – practical mathematics proper was thus certainly present throughout the third millennium.

This, however, is not yet the mathematics of RMP and MMP. Until Dynasty 5 only metrological sub-units and the fractions $3''$, $2'$, and $3'$ (and

[21] Thus, e.g., Alexander Badawy, *Ancient Egyptian Architectural Design. A Study of the Harmonic System* (Berkeley and Los Angeles, CA: University of California Press, 1965), pp. 3f. and passim.

[22] See Gay Robins and Charles C. D. Shute, "Mathematical Bases of Ancient Egyptian Architecture and Graphic Art," *Historia Mathematica* 12 (1985), 107–22.

[23] For further references regarding the fables and their lack of foundation, see Gillings, *Mathematics*, pp. 237–9. A recent, very careful treatment is Roger Herz-Fischler, *The Shape of the Great Pyramid* (Waterloo, Ontario: Wilfrid Laurier University Press, 2000).

[24] See the translation of the Palermo Stone annals in Clagett, *Ancient Egyptian Science*, vol. 1, 67–95.

a particular sign × for $^1/_4$) were in use – cf. the traditional way the batter of pyramids is expressed in RMP, in contrast to that of a structure with no links to the Old Kingdom. The first place where *ro*-writings of $\overline{4}$, $\overline{5}$, and $\overline{6}$ turn up is the twenty-fourth-century Abū Sir papyri – but since $\overline{5}$ appears in the sum $\overline{5}\ \overline{5}$ (for $^2/_5$) it is clear that the later canon for using the aliquot parts did not yet exist.[25] In Middle Kingdom administrative papyri, on the other hand, even aliquot parts too small to be noticed in practice are used routinely (18$\overline{0}$ of a jug of beer! – in contrast, both 14÷10 and 16÷10 may quietly be equalled to 1 2' in economical papyri[26]). Since the mathematics of MMP and RMP is shaped as a coherent structure precisely by the use of aliquot parts and by the techniques for dealing with them, *Ancient Egyptian mathematics* seems to be a creation of the early Middle Kingdom, and to have been made immediately the fundament for the mathematical training of scribes.

Conversely, a new organization of scribal training may have been the motive force that transformed a bundle of mathematical techniques into a unified whole.[27] Old Kingdom scribes had been taught in an apprenticeship system, and not in a school; the first reference to a school postdates the collapse of the Old Kingdom. Early Middle Kingdom scribes, on the other hand, had gone to school. For purposes of practical computation, metrological sub-units are much more convenient than aliquot parts and their sums (just as decimal fractions are more convenient than ordinary fractions). The advantages of the aliquot parts will only stand out as such within a school context that has gained some autonomy from immediate practice:

– they permit exactness (and thus the teacher's decision whether "you have found it correctly," as written in MMP);
– they allow theoretical coordination (clearly appreciated by the author of RMP, where everything until number 34 is abstract and expressed in pure numbers and aliquot parts);
– their use permits (and asks for) the display of virtuosity.

If we compare Middle Kingdom mathematics with Old Babylonian mathematics (see chapter 3 in the present volume, "Mesopotamian Mathematics"), we shall find no systematic, openly supra-utilitarian pursuits similar to Babylonian second-degree algebra. Analogues of the "humanism" of the Old Babylonian scribe school are also absent from the Egyptian school texts that served to inculcate professional norms and pride in future scribes. As we

[25] Further references in J. Høyrup, "On Parts of Parts and Ascending Continued Fractions," *Centaurus* 33 (1990), 293–324, 310. On $\overline{5}\,\overline{5}$ in particular, see David P. Silberman, "Fractions in the Abu Sir Papyri," *Journal of Egyptian Archaeology* 61 (1975), 248–9.

[26] See Paul J. Frandsen, [Review of J. J. Janssen, *Commodity Prices from the Ramessid Period* (Leiden: Brill, 1975)], *Acta Orientalia* 40 (1979), 279–302, here p. 283.

[27] See Helmuth Brunner, *Altägyptische Erziehung* (Wiesbaden: Otto Harrassowitz, 1957), pp. 11–15; and John Wilson in Carl Kraeling and Robert McC. Adams (eds.), *City Invincible* (Chicago, IL: University of Chicago Press, 1960), p. 103.

see, however, the difference is not absolute, and even in Egypt the scribe school transmuted the knowledge and skills it had to impart. In one respect the impact of schooling was even stronger in Egypt than in Babylonia: the fundamental practical techniques created during Ur III (admittedly within the school) were only affected superficially by the new climate of the Old Babylonian scribe school; instead, "humanism" expressed itself in the grafting of an additional, supra-utilitarian discipline on the curriculum. In Egypt, the systematic use of aliquot parts (more supra-utilitarian than normally recognized) transformed even ordinary mathematical practice.

LINKS?

These similarities are evidently to be explained as parallel developments due to similar conditions, not as borrowings. On the general level, second-millennium Egyptian and Babylonian mathematics are wholly independent from each other. On the level of particulars, the occasional multiplication with a reciprocal in RMP has sometimes been seen as a borrowing, but the context where it occurs speaks against that assumption (cf. above). Only a single problem in RMP (*viz.* no 37) is certainly related to a Babylonian text:

> Go down I [a jug of unknown capacity] times 3 into the *ḥeḳat*-measure, 3′ of me is added to me, 3′ of 3′ of me is added to me, $\overline{9}$ of me is added to me; return I, filled am I.

This can be compared with a problem from Old Babylonian Ešnunna:[28]

> To $^{2}/_{3}$ of my $^{2}/_{3}$ I have joined 100 sìla and my $^{2}/_{3}$, 1 gur was completed. The *tallum*-vessel of my grain corresponds to what?

The Egyptian solution is quite regular, fully based on aliquot parts and grain metrology; the Babylonian solution is no solution at all but a trick which presupposes the solution to be already known. The problem is obviously one of those riddles which the early Akkadian scribe school borrowed around 1800 BCE (see chapter 3 of the present volume). On the other hand, the idiom of "ascending continued fractions" ("*a*, and *b* of *a*," where *a* and *b* are simple fractions) is typically Semitic, and alien to the Egyptian context.[29] A Babylonian borrowing from Egypt as well as an Egyptian adoption of a Babylonian school problem are thus excluded; both must build on

[28] IM 53 957, edited in Taha Baqir, "Some More Mathematical Texts from Tell Harmal," *Sumer* 7 (1951), 28–45, 37, corrections and interpretation in W. von Soden, "Zu den mathematischen Aufgabentexten vom Tell Harmal," *Sumer* 8 (1952), 49–56, 52.

[29] See in general Høyrup, "On Parts of Parts and Ascending Continued Fractions"; since I had not noticed the Ešhnunna parallel at the time, this publication contains some speculations about a possible common Hamito-Semitic language structure; they may now be happily dismissed.

a common source, probably a traders' environment in contact with both regions. In contrast to what happened in Babylonia, however, such borrowings are not likely to have had any deeper influence on Egyptian Middle Kingdom mathematics, which instead developed material and ideas already present in third-millennium scribal computation.

DEMOTIC CREATIVITY AND BORROWINGS

Autochthonous (but after the Middle Kingdom maturation very slow) development remains a characteristic of the Egyptian mathematical tradition into the demotic period – and even into the early Byzantine epoch, as revealed by the Akhmīm-papyrus (written in Greek).[30] Development *does* take place: one undated demotic papyrus[31] tabulates $p{\cdot}\overline{q}$, $1 \leq p \leq 10$, $q = 90$ and $q = 150$ (similar tables are found in the Akhmīm papyrus; a modest beginning in RMP lists $p{\cdot}\overline{q}$, $1 \leq p \leq 10$, $q = 10$, but the context here suggests that other q-values would not be considered); the demotic papyri also transform the old technique of the reference quantity so as to express occasionally proper fractions, treating (e.g.) $5\ \underline{11}$ ("5 seen as part of 11") as a legitimate final result and not as a problem whose solution is $3'\ \overline{11}\ \overline{33}$; what happens can be characterized as a process of "creative dissolution" of the old canon which does not bring about any new coherence – a close parallel to the changes in the character of the visual arts in Hellenistic Egypt.

In this phase, however, influence from Western Asia is strong – no wonder, given that Egypt had been regularly controlled by Assyrian, Achaemenid, and Greek armies and tax-officials since the seventh century. These contacts (and perhaps trading connections) are likely to explain the use of a variety of formulae which have no Egyptian antecedents but coincide precisely or almost with formulae that were in constant use in Mesopotamia since the third or even fourth millennium: the determination of the circular area as $^3/_4$ of the squared diameter; computation of the volume of a truncated cone as the height times the mid-cross-section;[32] and the use of the "surveyors formula" (average length times average width) to calculate the areas of approximately rectangular quadrangles. Contact is certainly the reason that 8 of 40 problems in P. Cairo *J. E. 89127B30, 89137B43* belong to a characteristic Babylonian type (a reed first standing vertically against

[30] Edition with translation and commentary in J. Baillet, *Le Papyrus mathématique d'Akhmîm* (Mission Archéologique Française au Caire, Mémoires 9, 1; Paris: Leroux, 1892).

[31] P. British Museum 10794, published in Parker (ed.), *Demotic Mathematical Papyri*, pp. 72 f.

[32] The details are of some interest: the surface of the mid-cross-section is not found as in the corresponding Old Babylonian text, but as 1/4 of the product of diameter and arc, the arc being 3 diameters – see Clagett, *Ancient Egyptian Science.* Volume II*: Calendars, Clocks and Astronomy* (Memoirs of the American Philosophical Society, 214; Philadelphia, PA: American Philosophical Society, 1989), p. 75. The latter formula belongs to the lay tradition, is found in one Old Babylonian school tablet (dealing with a semicircle), and recurs in the pseudo-Heronic material.

a wall and then moved to a slanted position), which always involves the Pythagorean theorem, often in a sophisticated way (asking, e.g., for the legs of a right triangle when one leg and the difference between the hypotenuse and the other leg are given).

INFLUENCE ON GREEK GEOMETRY?

From Herodotus onward, common Greek lore asserted that geometry was invented by the Egyptians (either, in agreement with etymology, for surveying and taxation purposes, or by the priests who had sufficient leisure for such concerns). Since Egyptian and Babylonian mathematics became known directly, historians of mathematics have been puzzled by this claim. There is no doubt that the Greeks took their way to deal with fractions from Egypt; the canonical system for pictorial representation certainly influenced sixth-century Greek sculptors, and similar architectural rules may be reflected in Vitruvius; but none of these have anything to do with Greek (theoretical) *geometry*. At least one strain in Greek geometry (the "metric geometry" of *Elements* II etc.), on the other hand, has striking structural similarities with Babylonian algebra (see chapter 3 of the preent volume "Mesopotamian Mathematics"). Before we dismiss the Greek account as pure legend we should take note that the Greeks would only encounter Egyptian geometric practice well after the arrival of Assyrian and Achaemenid surveyors, and that the borrowings into demotic mathematics concern geometry, in particular metrical geometry. It is not to be excluded that early Greek geometry was inspired by what the Greeks encountered in Egypt; if so, the Greeks will have had little chance to know that what they encountered was a fairly recent import there.